

Common Fixed Point Theorem for Weakly Subsequentially Continuous Generalized Contraction Maps with Rational Expression

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ABSTRACT: In this paper, we prove the existence of Common fixed point for two pairs of selfmappings satisfying generalized contractive condition through rational expression by using the weak subsequential continuity with compatibility of type(E) in metric spaces.

KEYWORDS: Common fixed point compatibility of type(E), subsequentially continuous, metric space.

I. INTRODUCTION

In non-linear functional analysis, fixed point theory is indispensable due to its wide application to non-linear sciences besides various research fields in mathematics. Banach contraction principle[1] has been generalized in various ways either by using contractive conditions or by imposing some additional conditions on the ambient spaces. In 1976, Jungck [4] introduced the notion of commuting mappings and established a common fixed theorem. Later, Sessa [13] introduced a weaker notion weakly commuting mappings. Jungck [5] gave a generalization to the last notions. In the direction of generalization of contraction conditions, in 1975 Dass and Gupta [3] established fixed point results using contraction conditions involving rational expressions.

II. PRELIMANARIES

We denote $\Psi = \{\psi/\psi: [0, \infty) \rightarrow [0, \infty) \text{ is non-decreasing, continuous and } \psi(x) = 0 \Leftrightarrow x = 0\}$
and $\Phi = \{\phi/\phi: [0, \infty) \rightarrow [0, \infty) \text{ is lower semi continuous and } \phi(x) = 0 \Leftrightarrow x = 0\}$.
The following theorem is due to Dass and Gupta [3]

2.1 Theorem[3]: Let (X, d) be a complete metric space and $T: X \rightarrow X$ a mapping such that there exist $\alpha, \beta > 0$ with $\alpha + \beta < 1$ satisfying

$$d(Tx, Ty) \leq \alpha d(x, y) + \beta \frac{d(y, Ty)[1 + d(x, Tx)]}{1 + d(x, y)}$$

for all $x, y \in X$. Then T has a unique fixed point.

We use the following definitions in our subsequence discussion.

2.2 Definition[5]: Let (X, d) be a metric space. Two selfmappings A and B on X are said to be compatible, if $\lim_{n \rightarrow \infty} d(ABx_n, BAx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} Bx_n = t$ for some $t \in X$.

It is clear that "commuting" implies "weakly commuting" implies "compatible".

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2.3 Definition[6]: Let (X, d) be a metric space. Two selfmappings A and B on X are said to be compatible mapping of type(A) if $\lim_{n \rightarrow \infty} d(ASx_n, S^2x_n) = 0$ and $\lim_{n \rightarrow \infty} d(SAx_n, A^2x_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} Bx_n = t$ for some $t \in X$.

2.4 Definition[10]: Let (X, d) be a metric space. Two selfmappings A and B on X are said to be compatible mapping of type (B) if

$$\lim_{n \rightarrow \infty} d(SAx_n, A^2x_n) \leq \frac{1}{2} [\lim_{n \rightarrow \infty} d(St, Ax_n) + \lim_{n \rightarrow \infty} d(St, S^2x_n)],$$

$$\lim_{n \rightarrow \infty} d(ASx_n, S^2x_n) \leq \frac{1}{2} [\lim_{n \rightarrow \infty} d(ASx_n, At) + \lim_{n \rightarrow \infty} d(At, A^2x_n)]$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} Bx_n = t$ for some $t \in X$.

2.5 Definition[9]: Let (X, d) be a metric space. Two selfmappings A and B on X are said to be compatible mapping of type(P) if $\lim_{n \rightarrow \infty} d(A^2x_n, S^2x_n) = 0$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} Bx_n = t$ for some $t \in X$.

2.6 Definition[11]: Let (X, d) be a metric space. Two selfmappings A and B on X are said to be compatible mapping of type(C) if

$$\lim_{n \rightarrow \infty} d(ASx_n, S^2x_n) \leq \frac{1}{2} [\lim_{n \rightarrow \infty} d(ASx_n, At) + \lim_{n \rightarrow \infty} d(At, S^2x_n) + \lim_{n \rightarrow \infty} d(At, A^2x_n)]$$

$$\lim_{n \rightarrow \infty} d(SAx_n, A^2x_n) \leq \frac{1}{2} [\lim_{n \rightarrow \infty} d(SAx_n, St) + \lim_{n \rightarrow \infty} d(St, S^2x_n) + \lim_{n \rightarrow \infty} d(St, A^2x_n)]$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} Sx_n = t$ for some $t \in X$.

Notice that, compatibility of type(A), compatibility of type(B) and compatibility of type(C) are equivalent under the continuity of A and S .

2.7 Definition[7]: Let A and S be selfmaps of a metric space (X, d) . The pair (A, S) is said to be weakly compatible, if they commute at their coincidence points.

i.e., $ASt = SAT$ whenever $At = Bt, t \in X$.

2.8 Definition[14]: Two selfmappings A and S of a metric space (X, d) are said to be compatibility of type(E), if

$\lim_{n \rightarrow \infty} S^2x_n = \lim_{n \rightarrow \infty} SAx_n = At$ and $\lim_{n \rightarrow \infty} A^2x_n = \lim_{n \rightarrow \infty} ASx_n = St$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} Sx_n = t$, for some $t \in X$.

2.9 Remark: If $At = St$, then compatibility of type(E) implies compatibility (compatibility of type(A), compatibility of type(B), compatibility of type(C), compatibility of type(P)), however the converse may not be true. Generally if (A, S) is compatibility of type (E) implies compatibility of type(B).

2.10 Definition[15]: Two selfmappings A and S of a metric space (X, d) are S -compatibility (A -compatibility) of type (E), if $\lim_{n \rightarrow \infty} S^2x_n = \lim_{n \rightarrow \infty} SAx_n = At$ (respectively $\lim_{n \rightarrow \infty} A^2x_n = \lim_{n \rightarrow \infty} ASx_n = St$) for some $t \in X$.

Notice that if A and S are compatible of type(E), then they are A -compatible and S -compatible of type(E), but the converse is not true. [See example 1[12]].

2.11 Definition[8]: Two selfmappings A and S of a metric space (X, d) are said to be reciprocally continuous, if $\lim_{n \rightarrow \infty} ASx_n = At$ and $\lim_{n \rightarrow \infty} SAx_n = St$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} Sx_n = t$, for some $t \in X$.

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In 2009, Bouhadjera and Godet Thobie[15] introduced the concept of sub sequentially continuity as follows.

2.12 Definition[2]: Two selfmappings A and S of a metric space (X, d) are said to be subsequentially continuous if there exists a sequence $\{x_n\}$ such that

$$\lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} Sx_n = t, \text{ for some } t \in X, \lim_{n \rightarrow \infty} ASx_n = At \text{ and } \lim_{n \rightarrow \infty} SAX_n = St.$$

It is clear that A and S are continuous or reciprocally continuous, then they are subsequentially continuous.

2.13 Definition[12]: Let A and S to be two selfmappings A and S of a metric space (X, d) . The pair (A, S) is said to be weakly subsequentially continuous (wsc for short) if there exists a sequence $\{x_n\}$ such that $\lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} Sx_n = t$, for some $t \in X$ and $\lim_{n \rightarrow \infty} ASx_n = At$ or $\lim_{n \rightarrow \infty} SAX_n = St$.

2.14 Definition[12]: A pair (A, S) of selfmappings is said to be S -subsequentially continuous if there exists a sequence $\{x_n\}$ such that $\lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} Sx_n = t$, for some $t \in X$ and $\lim_{n \rightarrow \infty} SAX_n = St$.

2.15 Definition[12]: A pair (A, S) of selfmappings is said to be A -subsequentially continuous if there exists a sequence $\{x_n\}$ such that $\lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} Sx_n = t$, for some $t \in X$ and $\lim_{n \rightarrow \infty} ASx_n = At$.

If the (A, S) is A -subsequentially continuous(or S -subsequentially continuous), then it is wsc. [see example 3[12]].

III. MAIN RESULT

The following is our main result.

3.1 THEOREM: Let (X, d) be a metric space and $A, B, S, T: X \rightarrow X$ are four selfmappings such that for all $x, y \in X$, $\psi \in \Psi$, $\phi \in \Phi$ satisfy

$$\psi(d(Sx, Ty)) \leq \psi(M(x, y)) - \phi(M(x, y)) \text{ -----} \tag{1}$$

where

$$M(x, y) = \max\{d(By, Ty), d(Sx, Ax), d(Ax, By), \frac{d(Ax, Ty)d(By, Sx)}{1 + d(Sx, Ty)}, \frac{d(Ax, Sx)d(By, Ty)}{1 + d(Ax, By)}\}$$

If the two pairs (A, S) and (B, T) are weakly subsequentially continuous (w.s.c) and compatible of type (E) , then A, B, S and T have a unique common fixed point in X .

Proof: Suppose (A, S) is wsc (suppose that A -subsequentially continuous), then there exists a sequence $\{x_n\}$ in X such that $\lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} Sx_n = z$, for some $z \in X$ and $\lim_{n \rightarrow \infty} ASx_n = Az$.----- (2)

Again A and S are compatible of type (E) , so

$$\begin{aligned} \lim_{n \rightarrow \infty} A^2x_n &= \lim_{n \rightarrow \infty} ASx_n = Sz \text{ and} \\ \lim_{n \rightarrow \infty} S^2x_n &= \lim_{n \rightarrow \infty} SAX_n = Az \text{ .-----} \end{aligned} \tag{3}$$

From (2) and (3) we have $Az = Sz$.

Since (B, T) is wsc (suppose that B -subsequentially continuous), there exists a sequence $\{y_n\}$ in X such that

$$\lim_{n \rightarrow \infty} By_n = \lim_{n \rightarrow \infty} Ty_n = z', \text{ for some } z' \in X \text{ and } \lim_{n \rightarrow \infty} BTy_n = Bz'. \text{ -----} \tag{4}$$

Also, the pair (B, T) is compatible for type (E) , we have

$$\begin{aligned} \lim_{n \rightarrow \infty} B^2y_n &= \lim_{n \rightarrow \infty} BTy_n = Tz' \text{ and} \\ \lim_{n \rightarrow \infty} T^2y_n &= \lim_{n \rightarrow \infty} TBy_n = Bz'. \text{ -----} \end{aligned} \tag{5}$$

From (4) and (5), we have $Bz' = Tz'$

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We prove $Az = Bz'$ If $Az \neq Bz'$, then

By (1), we get

$$\psi(d(Az, Bz')) = \psi(d(Sz, Tz')) \leq \psi(M(z, z')) - \phi(M(z, z'))$$

$$\text{Where } M(z, z') = \max \left\{ d(Az, Sz), d(Bz', Tz'), d(Az, Bz'), \frac{d(Az, Tz')d(Bz', Sz)}{1+d(Sz, Tz')}, \frac{d(Az, Sz)d(Bz', Tz')}{1+d(Az, Bz')} \right\}$$

$$= \max \left\{ 0, 0, d(Sz, Tz'), \frac{d(Sz, Tz')d(Tz', Az)}{1+d(Sz, Tz')} \right\}$$

Therefore,

$$\psi(d(Az, Bz')) \leq \psi(d(Sz, Tz')) - \phi(d(Sz, Tz'))$$

$$= \psi(d(Az, Bz')) - \phi(d(Az, Bz')) < \psi(d(Az, Bz')),$$

a contradiction.

Therefore, $Az = Sz = Bz' = Tz'$.

Next, we now prove that $z = Az$.

Suppose that $z \neq Az$.

Now, we consider

$$\psi(d(Sx_n, Tz') \leq \psi(M(x_n, z')) - \phi(M(x_n, z'))$$

$$\text{where } M(x_n, z') = \max \left\{ d(Ax_n, Sx_n), d(Bz', Tz'), d(Ax_n, Bz'), \frac{d(Ax_n, Tz')d(Bz', Sx_n)}{1+d(Sx_n, Tz')}, \frac{d(Ax_n, Sx_n)d(Bz', Tz')}{1+d(Ax_n, Bz')} \right\}.$$

Letting $n \rightarrow \infty$, we get

$$\psi(d(z, Az) \leq \psi(\max \{0, 0, d(z, Az), \frac{d(z, Az)d(Az, z)}{1+d(z, Az)}, 0\}) - \phi(\max \{0, 0, d(z, Az), \frac{d(z, Az)d(Az, z)}{1+d(z, Az)}, 0\}))$$

$$= \psi(d(z, Az)) - \phi(d(z, Az)) < \psi(d(z, Az))$$

a contradiction.

Hence $z = Az = Sz$.

Now we prove that $z = z'$

If $z \neq z'$, then

$$\psi(d(Sx_n, Ty_n) \leq \psi(M(x_n, y_n)) - \phi(M(x_n, y_n))$$

$$\text{Where } M(x_n, y_n) = \max \left\{ d(Ax_n, Sx_n), d(By_n, Ty_n), d(Ax_n, By_n), \frac{d(Ax_n, Ty_n)d(By_n, Sx_n)}{1+d(Sx_n, Ty_n)}, \frac{d(Ax_n, Sx_n)d(By_n, Ty_n)}{1+d(Ax_n, By_n)} \right\}$$

On letting $n \rightarrow \infty$, we get

$$\psi(d(z, z')) \leq \psi(d(z, z')) - \phi(d(z, z')) < \psi(d(z, z')),$$

a contradiction.

Hence $z = Az = Sz = Bz = Tz$.

Therefore z is a common fixed point of A, B, S and T .

Uniqueness:

Suppose that $t \neq z$ is another common fixed point of A, B, S and T .

$$\psi(d(t, z)) = \psi(d(St, Tz)) \leq \psi(M(t, z)) - \phi(M(t, z))$$

$$\text{where } M(t, z) = \max \left\{ d(At, St), d(Bz, Tz), d(At, Bz), \frac{d(At, Tz)d(Bz, St)}{(1+d(St, Tz))}, \frac{d(At, St)d(Bz, Tz)}{(1+d(At, Bz))} \right\}$$

$$= d(t, z)$$

$$\text{Therefore, } \psi(d(t, z)) \leq \psi(d(t, z)) - \phi(d(t, z)) < \psi(d(t, z)),$$

a contradiction.

Therefore, $t = z$.

Hence z is the unique common fixed point of A, B, S and T .

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IV. CONCLUSION

We prove the existence of Common fixed point for two pairs of selfmappings satisfying generalized contractive condition with rational expression by using the weak subsequential continuity with compatibility of type(E) in metric spaces.

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